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GROUP PROPERTIES OF 2-SUBMODELS FOR THE EVOLUTIONARY CLASS OF GAS-DYNAMIC EQUATIONS

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Invariant 2-submodels (submodels with two independent variables) of the evolutionary class are considered for the equations of gas dynamics with an equation of state of general form. Group analysis of these submodels is performed. Allowable operators and transformations of equivalence are indicated, and group classification is performed.

As is known, all invariant 2-submodels (submodels with two independent variables) of the equations of gas dynamics reduce to one of the two following systems: the system of equations of the evolutionary class (E class) or the system of equations of the stationary class (S class) [1, 2]. All such submodels are obtained for the two-parameter subgroups corresponding to the subalgebras $L_{2,l}$ from Table 6 of [3] (below the numbering from Tab. 6 is used in references to the corresponding submodel).

In the present work, the invariant submodels of the evolution class are analyzed within the framework of the SUBMODEL program. The operators allowed by the system of a submodel and equivalence transformations are indicated, and group classification is performed.

The equations of the evolutionary class (class E) are obtained from submodels 2.8, 2.9, 2.10, 2.20, 2.21, 2.22, 2.23, 2.24, 2.25, and 2.27 and have the form

$$\begin{gathered} U_t + UU_\xi + (b/R)P_\xi = a_1, \qquad V_t + UV_\xi = a_2, \qquad W_t + UW_\xi = a_3, \\ R_t + UR_\xi + RU_\xi = Ra_4, \qquad P_t + UP_\xi + A(R,P)U_\xi = A(R,P)a_4, \end{gathered} \tag{1}$$

where $b = b(t) > 0$, the functions $a_i = a_i(t,\xi,U,V,W)$ are linear or quadratic functions of the variables U, V, and W, $A = Rc^2 = -RS_R/S_P$, and the velocity of sound $c = c(R,P)$ is obtained from the equation of state $S = S(R,P)$.

The equation for entropy S becomes $S_t + US_\xi = 0$.

The invariant variables ξ, U, V, W, R, and P for each submodel are described in [2]. The choice of these variable is not unique. Thus, in submodel 2.8, instead of the functions V, W one can introduce the functions tV and ξW. In this case, the right sides of the equations a_2 and a_3 become zero. This, however, does not introduce a considerable simplification of the equations, and, therefore, in the present work, we use the variables described in [2].

The Lie algebra operators allowed by the system are sought in the form

$$X = \alpha^t\partial_t + \alpha^\xi\partial_\xi + \alpha^U\partial_U + \alpha^V\partial_V + \alpha^W\partial_W + \alpha^R\partial_R + \alpha^P\partial_P.$$

All coefficients α are functions of the variables t, ξ, U, V, W, R, and P. The extended operator $\tilde{X}$ is written as

$$\tilde{X} = X + \zeta^{U_t}\partial_{U_t} + \ldots + \zeta^{R_\xi}\partial_{R_\xi}.$$

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Here

$$\zeta^{U_t} = D_t(\alpha^U) - U_t D_t(\alpha^t) - U_\xi D_t(\alpha^\xi), \qquad \zeta^{U_\xi} = D_\xi(\alpha^U) - U_t D_\xi(\alpha^t) - U_\xi D_\xi(\alpha^\xi),$$

. . . .

$$\zeta^{R_t} = D_t(\alpha^R) - R_t D_t(\alpha^t) - R_\xi D_t(\alpha^\xi), \qquad \zeta^{R_\xi} = D_\xi(\alpha^R) - R_t D_\xi(\alpha^t) - R_\xi D_\xi(\alpha^\xi),$$

where

$$D_t = \partial_t + U_t\partial_U + V_t\partial_V + W_t\partial_W + P_t\partial_P + R_t\partial_R, \quad D_\xi = \partial_\xi + U_\xi\partial_U + V_\xi\partial_V + W_\xi\partial_W + P_\xi\partial_P + R_\xi\partial_R$$

(D_t and D_ξ are full differential operators).

Let us act on system (1) by the extended operator. After substitution of the expressions for the coefficients ζ and elimination of the derivatives U_t, V_t, W_t, R_t, and P_t, taken from system (1), we obtain five equalities. Setting the coefficients at the quadratic terms with derivatives of U_ξ, V_ξ, W_ξ R_ξ, and P_ξ equal to zero, we obtain 2-equations, and setting the coefficients of the linear terms equal to zero, we have 1-equations. The remaining terms lead to 0-equations.

From the 2-equations it only follows that there is x-autonomy (by the nomenclature of [4]): $\alpha^\xi = \alpha^\xi(t,\xi)$ and $\alpha^t = \alpha^t(t,\xi)$.

From an analysis of the 1-equations it follows that

$$\alpha^t = \alpha^t(t), \qquad \alpha^\xi = \alpha^\xi(t,\xi), \qquad \alpha^U = \alpha^\xi_t + U(\alpha^\xi_\xi - \alpha^t_t),$$

$$\alpha^V = \alpha^V(t,\xi,V,W,R,P), \qquad \alpha^W = \alpha^W(t,\xi,V,W,R,P),$$

$$\alpha^R = R\alpha^P_P + R(2\alpha^t_t - 2\alpha^\xi_\xi + (b_t/b)\alpha^t), \qquad \alpha^P = \alpha^P(t,\xi,P), \qquad A\alpha^P_{PP} = 0,$$

$$A_P\alpha^P + A_R\alpha^R - A\alpha^P_P = 0, \qquad A\alpha^V_P + R\alpha^V_R = 0, \qquad A\alpha^W_P + R\alpha^W_R = 0.$$

If $A = 0$, then $\alpha^P = \alpha^P(t,\xi,P)$, $\alpha^V = \alpha^V(t,\xi,V,W,P)$, and $\alpha^W = \alpha^W(t,\xi,V,W,P)$.

If $A \neq 0$, then $\alpha^P = f^P(t,\xi)P + g^P(t,\xi)$.

Instead of the variables R and P we introduce the variables R and S, where S is any function that satisfies the equation $S_t + US_\xi = 0$ (in particular, entropy). Then $\alpha^V_R = 0$ and $\alpha^W_R = 0$, whence $\alpha^V = \alpha^V(t,\xi,V,W,S)$ and $\alpha^W = \alpha^W(t,\xi,V,W,S)$.

The 0-equations have the form

$$\begin{gathered}
(b/R)\alpha^P_\xi + U\alpha^U_\xi - a_1\alpha^t_t + \alpha^U_t + a_1\alpha^U_U - (a_{1t}\alpha^t + a_{1\xi}\alpha^\xi + a_{1U}\alpha^U + a_{1V}\alpha^V + a_{1W}\alpha^W) = 0,\\
\alpha^V_t - a_2\alpha^t_t + U\alpha^V_\xi + a_2\alpha^V_V + a_3\alpha^V_W - (a_{2t}\alpha^t + a_{2\xi}\alpha^\xi + a_{2U}\alpha^U + a_{2V}\alpha^V + a_{2W}\alpha^W) = 0,\\
\alpha^W_t - a_3\alpha^t_t + U\alpha^W_\xi + a_2\alpha^W_V + a_3\alpha^W_W - (a_{3t}\alpha^t + a_{3\xi}\alpha^\xi + a_{3U}\alpha^U + a_{3V}\alpha^V + a_{3W}\alpha^W) = 0,\\
\alpha^R_t - Ra_4\alpha^t_t + U\alpha^R_\xi + R\alpha^U_\xi - R(a_{4t}\alpha^t + a_{4\xi}\alpha^\xi + a_{4U}\alpha^U + a_{4V}\alpha^V + a_{4W}\alpha^W) = 0,\\
\alpha^P_t + U\alpha^P_\xi - Aa_4\alpha^t_t + A\alpha^U_\xi - A(a_{4t}\alpha^t + a_{4\xi}\alpha^\xi + a_{4U}\alpha^U + a_{4V}\alpha^V + a_{4W}\alpha^W) = 0.
\end{gathered} \tag{2}$$

To obtain equivalence transformations, we supplement the initial system (1) with the equations $A_t = A_\xi = A_U = A_V = A_W = 0$. The equivalence transformations operators are sought in the form

$$X^e = \alpha^t\partial_t + \alpha^\xi\partial_\xi + \alpha^U\partial_U + \alpha^V\partial_V + \alpha^W\partial_W + \alpha^R\partial_R + \alpha^P\partial_P + \alpha^A\partial_A.$$

All coefficients α are functions of the variables t, ξ, U, V, W, R, P, and A.

Let us introduce the full differential operators

$$D^e_t = \partial_t + U_t\partial_U + V_t\partial_V + \ldots + (A_R R_t + A_P P_t)\partial_A,$$

$$D^e_\xi = \partial_\xi + U_\xi \partial_U + V_\xi \partial_V + \ldots + (A_R R_\xi + A_P P_\xi)\partial_A,$$

$$\widetilde{D^e_t} = \partial_t, \qquad \widetilde{D^e_\xi} = \partial_\xi, \qquad \widetilde{D^e_U} = \partial_U, \qquad \widetilde{D^e_V} = \partial_V,$$

$$\widetilde{D^e_W} = \partial_W, \qquad \widetilde{D^e_R} = \partial_R + A_R\partial_A, \qquad \widetilde{D^e_P} = \partial_P + A_P\partial_A.$$

The coefficients of the extended operator

$$\widetilde{X^e} = X^e + \zeta^{U_t}\partial_{U_t} + \ldots + \zeta^{R_\xi}\partial_{R_\xi} + \zeta^{A_t}\partial_{A_t} + \ldots + \zeta^{A_R}\partial_{A_R}$$

are obtained form the formulas

$$\zeta^{U_t} = D^e_t\alpha^U - U_t D^e_t(\alpha^t) - U_\xi D^e_t(\alpha^\xi),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\zeta^{R_\xi} = D^e_\xi\alpha^R - R_t D^e_\xi(\alpha^t) - R_\xi D^e_\xi(\alpha^\xi),$$

$$\zeta^{A_t} = \widetilde{D^e_t}\alpha^A - A_R\widetilde{D^e_t}(\alpha^R) - A_P\widetilde{D^e_t}(\alpha^P),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\zeta^{A_P} = \widetilde{D^e_P}\alpha^A - A_R\widetilde{D^e_P}(\alpha^R) - A_P\widetilde{D^e_P}(\alpha^P).$$

Let us act on the system by the extended operator. After elimination of the derivatives U_t, V_t, W_t, R_t, and P_t, we have 10 equalities. Setting the coefficients of the cubic terms with derivatives of U_ξ, V_ξ, W_ξ, R_ξ, P_ξ, A_R, and A_P equal to zero, we obtain 3-equations, and setting the coefficients of the quadratic and linear terms to zero, we obtain 2-equations and 1-equations, respectively. The remaining terms give 0-equations.

From the 3- and 2-equations it follows that

$$\alpha^t = \alpha^t(t,\xi), \quad \alpha^\xi = \alpha^\xi(t,\xi), \quad \alpha^U = \alpha^U(t,\xi,U,V,W,R,P), \quad \alpha^V = \alpha^V(t,\xi,U,V,W,R,P),$$

$$\alpha^W = \alpha^W(t,\xi,U,V,W,R,P), \qquad \alpha^R = \alpha^R(t,\xi,U,V,W,R,P),$$

$$\alpha^P = \alpha^P(t,\xi,U,V,W,R,P), \qquad \alpha^A = \alpha^A(R,P,A).$$

An analysis of the 1- and 0-equations shows that

$$\alpha^t = \alpha^t(t), \quad \alpha^\xi = \alpha^\xi(t,\xi), \quad \alpha^U = \alpha^\xi_t + U(\alpha^\xi_\xi - \alpha^t_t), \quad \alpha^V = \alpha^V(t,\xi,V,W),$$

$$\alpha^W = \alpha^W(t,\xi,V,W), \quad \alpha^R = C_1R + 2R(\alpha^t_t - \alpha^\xi_\xi) + R(b_t/b)\alpha^t, \quad \alpha^P = C_1P + C_2, \quad \alpha^A = C_1A.$$

The remaining 0-equations become

$$\begin{gathered}
\alpha^U_t - a_1\alpha^t_t + U\alpha^U_\xi + a_1\alpha^U_U - a_{1t}\alpha^t - a_{1\xi}\alpha^\xi - a_{1U}\alpha^U - a_{1V}\alpha^V - a_{1W}\alpha^W = 0,\\
\alpha^V_t + U\alpha^V_\xi + a_2\alpha^V_V + a_3\alpha^V_W - a_2\alpha^t_t - a_{2t}\alpha^t - a_{2\xi}\alpha^\xi - a_{2U}\alpha^U - a_{2V}\alpha^V - a_{2W}\alpha^W = 0,\\
\alpha^W_t + U\alpha^W_\xi + a_2\alpha^W_V + a_3\alpha^W_W - a_3\alpha^t_t - a_{3t}\alpha^t - a_{3\xi}\alpha^\xi - a_{3U}\alpha^U - a_{3V}\alpha^V - a_{3W}\alpha^W = 0,\\
\alpha^U_\xi - a_4\alpha^t_t - (a_{4t}\alpha^t + a_{4\xi}\alpha^\xi + a_{4U}\alpha^U + a_{4V}\alpha^V + a_{4W}\alpha^W) = 0.
\end{gathered} \tag{3}$$

Let us analyze the 0-equations of (2) and (3) for each submodel. To describe the results, we introduce the operators

$$Z_1 = \partial_t, \quad Z_2 = \partial_\xi, \quad Z_3 = t\partial_t + \xi\partial_\xi, \quad Z_4 = t\partial_t + \xi\partial_\xi - V\partial_V, \quad Z_5 = t\partial_\xi + \partial_U,$$

$$Z_6 = t\partial_t + \xi\partial_\xi - tW\partial_V, \quad Z_7 = t\partial_\xi + \partial_U + \alpha\partial_V + t\partial_W, \quad Z_8 = (\alpha^2+1)\partial_V + \alpha t\partial_W,$$

$$Z_9 = t^2\partial_\xi + 2t\partial_U + \alpha t\partial_V + (t^2-1)\partial_W, \quad Z_{10} = \lambda(V,W)\,\partial_V, \quad Z_{10}^* = \partial_V, \quad Z_{10}^{**} = W\partial_V,$$

$$Z_{11} = \mu(V,W)\,\partial_W, \quad Z_{11}^* = \partial_W, \quad Z_{11}^{**} = V\partial_W, \quad Z_{12} = f(\xi W)\partial_V, \quad Z_{12}^* = \partial_V,$$

$$Z_{13} = \lambda(tW+V,W)\,\partial_V, \quad Z_{14} = \mu(tW+V,W)(t\,\partial_V - \partial_W), \quad Z_{14}^* = t\partial_V - \partial_W,$$

$$Z_{15} = (1/t)\lambda(tV,tW)\,\partial_V, \quad Z_{16} = (1/t)\mu(tV,tW)\,\partial_W, \quad Z_{17} = (1/t)\lambda(tV-\alpha tW,W)\,\partial_V,$$

$$Z_{18} = \mu(tV-\alpha tW,W)(\alpha\partial_V + \partial_W), \quad Z_{19} = (1/t)f(\xi W)\,\partial_V, \quad Z_{19}^* = (1/t)\,\partial_V,$$

$$Z_{20} = (1/t)f(tV,\xi W)\,\partial_V, \quad Z_{21} = t\partial_\xi + \partial_U + \frac{\alpha t - \beta\sigma}{t^2-\sigma\tau}\,\partial_V + \frac{\beta t - \alpha\tau}{t^2-\sigma\tau}\,\partial_W,$$

$$Z_{22} = \frac{\alpha\tau - \beta t}{t^2-\sigma\tau}\,\partial_\xi + \frac{\tau(\beta\sigma-\alpha t) - t(\alpha\tau-\beta t)}{(t^2-\sigma\tau)^2}\partial_U + \frac{\tau}{t^2-\sigma\tau}\,\partial_V - \frac{t}{t^2-\sigma\tau}\,\partial_W,$$

$$Z_{23} = \frac{\beta\sigma - \alpha t}{t^2-\sigma\tau}\,\partial_\xi + \frac{\sigma(\alpha\tau-\beta t) - t(\beta\sigma-\alpha t)}{(t^2-\sigma\tau)^2}\,\partial_U - \frac{t}{t^2-\sigma\tau}\,\partial_V + \frac{\sigma}{t^2-\sigma\tau}\,\partial_W,$$

$$Z_{24} = \frac{\beta\sigma - \alpha t}{t^2-\sigma\tau}\,t\partial_\xi - \frac{\sigma(\beta t^2 - 2\alpha\tau t + \beta\sigma\tau)}{(t^2-\sigma\tau)^2}\,\partial_U - \frac{\sigma\tau}{t^2-\sigma\tau}\,\partial_V + \frac{\sigma t}{t^2-\sigma\tau}\,\partial_W,$$

$$Z_{25} = \frac{\alpha\tau - \beta t}{t^2-\sigma\tau}\,t\partial_\xi - \frac{\tau(\alpha t^2 - 2\beta\sigma t + \alpha\sigma\tau)}{(t^2-\sigma\tau)^2}\,\partial_U + \frac{\tau\, t}{t^2-\sigma\tau}\,\partial_V - \frac{\sigma\tau}{t^2-\sigma\tau}\,\partial_W,$$

which belong to the kernels of the allowed groups, and the "extending" operators

$$Y_1 = R\partial_R + P\partial_P, \quad Y_2 = \partial_P, \quad Y_3 = -t\partial_t + U\partial_U - 2R\partial_R, \quad Y_4 = \xi\partial_\xi + U\partial_U - 2R\partial_R,$$

$$Y_5 = \xi\partial_\xi + U\partial_U + W\partial_W - 2R\partial_R, \qquad Y_6 = -t\partial_t + U\partial_U + tW\partial_V - 2R\partial_R,$$

$$Y_7 = t\partial_t + \xi\partial_\xi + 2/(1+t^2)(R\partial_R + P\partial_P), \quad Y_8 = t^2\partial_t + t\xi\partial_\xi + (\xi - tU)\partial_U - 3tP\partial_P - tR\partial_R,$$

$$Y_9 = t^2\partial_t + t\xi\partial_\xi + (\xi - tU)\partial_U - t^2W\partial_V - 3tP\partial_P - tR\partial_R,$$

$$Y_{10} = t^2\partial_t + t\xi\partial_\xi + (\xi - tU)\partial_U - t(V-\alpha W)\partial_V - 4tP\partial_P - 2tR\partial_R,$$

$$Y_{11} = \partial_t - (1/t)((V-\alpha W)\,\partial_W + R\partial_R + P\partial_P),$$

$$Y_{12} = t^2\partial_t + t\xi\partial_\xi + (\xi - tU)\,\partial_U - tW\partial_W - 2tR\partial_R - 4tP\partial_P,$$

$$Y_{13} = t^2\partial_t + t\xi\partial_\xi + (\xi - tU)\,\partial_U - tV\partial_V - tW\partial_W - 3tR\partial_R - 5tP\partial_P,$$

$$Y_{14} = (t^2+1)\,\partial_t + t\xi\partial_\xi + (\xi - tU)\,\partial_U - 3tR\partial_R - 5tP\partial_P,$$

$$Y_{15} = \partial_t - 2t/(1+t^2)(R\partial_R + P\partial_P), \qquad Y_{16} = \partial_t - (1/t)(V\partial_V + R\partial_R + P\partial_P),$$

$$Y_{17} = \partial_t - (1/t)(V\partial_V + W\partial_W + 2R\partial_R + 2P\partial_P), \qquad Y_g = Rg'(P)\,\partial_R + g(P)\,\partial_P.$$

Here α, β, σ, and τ are parameters of the subalgebra series and f, g, λ, and μ are arbitrary functions.

The kernels of the allowable algebras (intersection of the algebras allowed by systems with different functions A) are given in Table 1. It is known that the kernels contain normalizer factors, which are computed directly from the subalgebras generating the submodel [3]. The extension of the normalizer factor in a kernel is indicated in last column of Table 1.

For special equations of state, extensions of the kernel of the allowable algebras are possible. Tables 2 and 3 give all possible extensions (F is an arbitrary function).

TABLE 1

Submodel	Normalizer factor	Supplementary operators
2.8	Z_3, Z_{19}^*	Z_{20}
2.9	Z_1, Z_3, Z_{12}^*	Z_{12}
2.10	Z_4, Z_{19}^*	Z_{19}
2.20	Z_2, Z_{21}, Z_{22}, Z_{23}, Z_{24}, Z_{25}	—
2.21	Z_2, Z_5	Z_{10}, Z_{11}
2.22	Z_2, Z_3, Z_5, Z_{19}^*, Z_{16}^*, $Z_{10}^{**} - Z_{11}^{**}$	Z_{15}, Z_{16}
2.23	Z_2, Z_7, Z_8, Z_9	—
2.24	Z_2, Z_3, Z_5, $\alpha Z_{12}^* + Z_{11}^*$, Z_{19}^*	Z_{17}, Z_{18}
2.25	Z_1, Z_2, Z_5, Z_{12}^*, Z_{14}^*	Z_6, Z_{13}, Z_{14}
2.27	Z_1, Z_2, Z_3, Z_5, Z_{10}^*, Z_{11}^*, $Z_{10}^{**} - Z_{11}^{**}$	Z_{10}, Z_{11}

For completeness of the consideration, we give:

1) the coefficients of the submodels considered;

2) equivalence transformations for them.

Submodel 2.8

1) $b = 1$, $a_1 = W^2/\xi$, $a_2 = -V/t$, $a_3 = -UW/\xi$, and $a_4 = -(1/t + U/\xi)$;

2) $\xi^* = q_1\xi$, $U^* = q_1U$, $W^* = q_1W$, $P^* = q_2q_1^2(P + q_3)$, $R^* = q_2R$, and $A^* = q_2q_1^2A$.

Submodel 2.9

1) $b = 1$, $a_1 = W^2/\xi$, $a_2 = -\beta W/\xi$, $a_3 = -UW/\xi$, and $a_4 = -U/\xi$;

2) $\xi^* = q_1\xi$, $U^* = q_1U$, $V^* = q_4V + q_5h(\xi W, \beta)$, $W^* = q_1W$, $P^* = q_2P + q_3$, $R^* = q_1^{-2}q_2R$, $A^* = q_2A$, and $\beta^* = q_4\beta$ (h is an arbitrary function).

Submodel 2.10

1) $b = 1$, $a_1 = W^2/\xi$, $a_2 = W/(t\xi) - V/t$, $a_3 = -UW/\xi$, and $a_4 = -(1/t + U/\xi)$;

2) $\xi^* = q_1\xi$, $U^* = q_1U$, $W^* = q_1W$, $R^* = q_2R$, $P^* = q_2q_1^2(P + q_3)$, and $A^* = q_2q_1^2A$.

Submodel 2.20

1) $b = 1 + ((\beta\sigma - \alpha t)/(t^2 - \sigma\tau))^2 + ((\alpha\tau - \beta t)/(t^2 - \sigma\tau))^2$, $a_1 = (2/\Delta)[(-(\alpha^2 + \beta^2)t^3 3 + \alpha\beta(\sigma + \tau)t^2 - (2\beta^2\sigma^2 + 2\alpha^2\tau^2 + \sigma\tau(\alpha^2+\beta^2)t + \alpha\beta\sigma\tau(\sigma+\tau)))U + (\alpha t^4 - 2\beta\sigma t^3 + \beta(-\beta^2\sigma + 2\sigma^2\tau + \alpha^2\tau)t + \alpha(\beta^2\sigma\tau - \alpha^2\tau^2 - \sigma^2\tau^2))V + (\beta t^4 - 2\alpha\tau t^3 + \alpha(2\sigma\tau^2 + \beta^2\sigma - \alpha^2\tau)t + \beta\sigma(\alpha^2\tau - \beta^2\sigma - \sigma\tau^2))W]$, $a_2 = (1/\Delta)[((t^2 - \sigma\tau)(\tau - \sigma)(\alpha\tau - \beta t))U - (t^5 + (\alpha^2 + \beta^2 - 2\sigma\tau)t^3 - \alpha\beta(3\sigma + \tau)t^2 + \sigma(2\beta^2\sigma + \sigma\tau^2 + \alpha^2\tau - \beta^2\tau)t + \alpha\beta\sigma\tau(\sigma - \tau))V + (\tau t^4 + (\alpha^2\tau + \beta^2\sigma - 2\sigma\tau^2)t^2 - 4\alpha\beta\sigma\tau t + \sigma\tau(\sigma\tau^2 + \alpha^2\tau + \beta^2\sigma))W]$, $a_3 = (1/\Delta)[(t^2 - \sigma\tau)(\alpha t - \beta\sigma)(\tau - \sigma)U + (\sigma t^4 + (\beta^2\sigma + \alpha^2\tau - 2\sigma^2\tau)t^2 - 4\alpha\beta\sigma\tau t + \sigma\tau(\alpha^2\tau + \sigma^2\tau + \beta^2\sigma))V - (t^5 + (\alpha^2 + \beta^2 - 2\sigma\tau)t^3 - \alpha\beta(\sigma + 3\tau)t^2 + \tau(2\alpha^2\tau + \sigma^2\tau - \alpha^2\sigma + \beta^2\sigma)t + \alpha\beta\sigma\tau(\sigma - \tau))W]$, $a_4 = -2t/(t^2 - \sigma\tau)$, and $\Delta = (t^2 - \sigma\tau)[(t^2 - \sigma\tau)^2 + (\alpha\tau - \beta t)^2 + (\alpha t - \beta\sigma)^2]$;

2) $P^* = q_1P + q_2$, $R^* = q_1R$, and $A^* = q_1A$.

Submodel 2.21

1) $b = 1$, $a_1 = a_2 = a_3 = 0$, and $a_4 = -2t/(1 + t^2)$;

2) $P^* = q_3P + q_2$, $R^* = q_1q_3R$, and $A^* = q_3A$.

TABLE 2

A	Submodel		
	2.8, 2.10	2.9	2.20, 2.23
$PF(P\,R^{-\gamma})$	$(\gamma-1)Y_5+2\gamma Y_1$	$(\gamma-1)Y_5+2\gamma Y_1$	—
$PF(P\,R^{-1})$	Y_1	Y_1	Y_1
$F(P)$	Y_5	Y_5	—
$PF(R)$	$2Y_1+Y_5$	$2Y_1+Y_5$	—
γP	$Y_1,\ Y_5$	$Y_1,\ Y_5$	—
P	$Y_1,\ Y_5,\ Y_{16}$	—	—
$(5/3)P$	$Y_1,\ Y_5,\ Y_{13}$	—	—
$2P$	—	$Y_1,\ Y_5,\ Y_{12}$	—
$3P$	—	—	—
$F(R\,\mathrm{e}^{-P})$	Y_5-2Y_2	Y_5-2Y_2	—
$F(R)$	Y_2	Y_2	Y_2
γR^{γ}	$Y_2,(\gamma-1)Y_5+2\gamma Y_1$	$Y_2,(\gamma-1)Y_5+2\gamma Y_1$	Y_2
R	$Y_1,\ Y_2$	$Y_1,\ Y_2$	$Y_1,\ Y_2$
1	$Y_2,\ Y_5$	$Y_2,\ Y_5$	Y_2
0	$Y_5,\ Y_g$	$Y_5,\ Y_g$	Y_g

TABLE 3

A	Submodel				
	2.21	2.22	2.24	2.25	2.27
$PF(P\,R^{-\gamma})$	$(\gamma-1)Y_4+$ $+2\gamma Y_1$	$(\gamma-1)Y_4+$ $+2\gamma Y_1$	$(\gamma-1)Y_4+$ $+2\gamma Y_1$	$(\gamma-1)Y_6+$ $+2\gamma Y_1$	$(\gamma-1)Y_3+$ $+2\gamma Y_1$
$F(P)$	Y_4	Y_4	Y_4	Y_6	Y_3
$PF(R)$	$2Y_1+Y_4$	$2Y_1+Y_4$	$2Y_1+Y_4$	$2Y_1+Y_6$	$2Y_1+Y_3$
γP	$Y_1,\ Y_4$	$Y_1,\ Y_4$	$Y_1,\ Y_4$	$Y_1,\ Y_6$	$Y_1,\ Y_3$
P	$Y_1,\ Y_4,\ Y_7,\ Y_{15}$	$Y_1,\ Y_4,\ Y_{17}$	$Y_1,\ Y_4,\ Y_{11}$	—	—
$(5/3)P$	$Y_1,\ Y_4,\ Y_{14}$	$Y_1,\ Y_4,\ Y_{13}$	—	—	—
$2P$	—	—	$Y_1,\ Y_4,\ Y_{10}$	—	—
$3P$	—	—	—	$Y_1,\ Y_6,\ Y_9$	$Y_1,\ Y_3,\ Y_8$
$F(R\,\mathrm{e}^{-P})$	Y_4-2Y_2	Y_4-2Y_2	Y_4-2Y_2	Y_6-2Y_2	Y_3-2Y_2
$F(R)$	Y_2	Y_2	Y_2	Y_2	Y_2
γR^{γ}	$Y_2,(\gamma-1)Y_4+$ $+2\gamma Y_1$	$Y_2,(\gamma-1)Y_4+$ $+2\gamma Y_1$	$Y_2,(\gamma-1)Y_4+$ $+2\gamma Y_1$	$Y_2,(\gamma-1)Y_6+$ $+2\gamma Y_1$	$Y_2,(\gamma-1)Y_3+$ $+2\gamma Y_1$
R	$Y_1,\ Y_2$	$Y_1,\ Y_2$	$Y_1,\ Y_2$	$Y_1,\ Y_2$	$Y_1,\ Y_2$
1	$Y_2,\ Y_4$	$Y_2,\ Y_4$	$Y_2,\ Y_4$	$Y_2,\ Y_6$	$Y_2,\ Y_3$
0	$Y_4,\ Y_g$	$Y_4,\ Y_g$	$Y_4,\ Y_g$	$Y_6,\ Y_g$	$Y_3,\ Y_g$

Submodel 2.22

1) $b = 1$, $a_1 = 0$, $a_2 = -V/t$, $a_3 = -W/t$, and $a_4 = -2/t$;
2) $\xi^* = q_1\xi$, $U^* = q_1 U$, $P^* = q_2 q_1^2 (P + q_3)$, $R^* = q_2 R$, and $A^* = q_2 q_1^2 A$.

Submodel 2.23

1) $b = 1 + \alpha^2 + t^2$, $a_1 = 2(tU + \alpha tV - (1 + \alpha^2)W)/(1 + \alpha^2 + t^2)$, $a_2 = \alpha(tU + \alpha tV - (1 + \alpha^2)W)/(1 + \alpha^2 + t^2)$, $a_3 = ((1 + t^2)U + \alpha(1 + t^2)V - \alpha^2 tW)/(1 + \alpha^2 + t^2)$, and $a_4 = 0$;
2) $P^* = q_1 P + q_2$, $R^* = q_1 R$, and $A^* = q_1 A$.

Submodel 2.24

1) $b = 1$, $a_1 = 0$, $a_2 = -V/t + \alpha W/t$, $a_3 = 0$, and $a_4 = -1/t$;
2) $\xi^* = q_1\xi$, $U^* = q_1 U$, $R^* = q_2 R$, $P^* = q_2 q_1^2 (P + q_3)$, and $A^* = q_1^2 A$.

Submodel 2.25

1) $b = 1$, $a_1 = 0$, $a_2 = -W$, and $a_3 = a_4 = 0$;
2) $t^* = q_1 t$, $U^* = q_1^{-1} U$, $W^* = q_1^{-1} W$, $R^* = (q_1 + q_2)R$, $P^* = q_2 P + q_3$, and $A^* = q_2 A$.

Submodel 2.27

1) $b = 1$ and $a_1 = a_2 = a_3 = a_4 = 0$;
2) $t^* = q_1^{-1} t$, $U^* = q_1 U$, $R^* = q_2 R$, $P^* = (q_1^2 + q_2)P$, and $A^* = (q_1^2 + q_2)A$.

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